PSL WEEK 2025 — MULTIPARAMETER PERSISTENT HOMOLOGY

ALEX FERNANDES

So far, persistent modules were introduced in the one-parameter case, namely as collections of finite dimensional vector spaces indexed by a totally ordered set. Here, we will will generalise this point of view by introducing multi-parameter persistence i.e. by authorising partially ordered set of indices.

Multiparameter persistence modules

Let $n \in \mathbb{N}$, the number of direction in which we aim to filter, moreover, let $\mathbf{k}$ denote generically the field $\mathbb{R}$ or the field $\mathbb{Z}_2$.

First of all, we need to introduce a poset that realise the motivation of two axis persistence modules. Let $\mathcal{P}$ denote generically the poset $(\mathbb{R}^n, \leq^n)$ where $\leq^n$ is the product order, namely the order defined by

$$(x_1, \ldots, x_n) \leq^n (y_1, \ldots, y_n) \text{ iff } \forall i \in [n] : x_i \leq y_i.$$

Remark 1. The product order is linked with the cone $\gamma = [0, +\infty)^n$ as $x \leq^n y$ if and only if $y + \gamma \subseteq x + \gamma$. This definition is helpful whenever one aims to use the geometric point of view on persistence [6].

In order to integrate our newfangled order into persistence theory one needs to enlarge the previously seen definition of persistence module.

Definition 1. A *n-parameter persistence module* M on $\mathcal{P}$ is the data of a collection of finite dimensional vector space[1] $(M_x)_{x\in\mathcal{P}}$ and a collection of linear maps

$$\{M_x^y : M_x \to M_y \mid \forall x, y \in \mathcal{P} : x \leq y\}.$$

We also requires the following compatibility property, for any x, y, z, t such that $x \leq y \leq t$ and $x \leq z \leq t$ one has $M_x^y M_y^t = M_x^z M_z^t$.

An example of persistence module is the *interval module* given for any interval $A \subset \mathcal{P}$ by the one-dimensional vector spaces $\mathbf{k}$ whenever $x \in A$ and the zero dimensional vector space otherwise and the linear maps are given by the identity whenever it is possible. We shall denote this persistence module $\mathbf{k}^A$.

Multi-parameter persistence module appear naturally in the following context. Suppose that X is a topological space of interest[2] and let $f : X \to \mathbb{R}^n$ a feature map.

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[1]This condition is not necessarily required, see pointwise finite dimensional persistence module [8]

[2]that is reasonable, say a smooth-submanifold of $\mathbb{R}^n$ for instance

Definition 2. For any $d \in \mathbb{N}$ define the f-sublevel set persistent homology of degree d as the n-parameter persistence module given by

$$H_d[f] : a \mapsto H_d(\{x \in X \mid f(x) \leq a\})$$

with linear maps induced by the filtration.

For sake of simplicity, we denote $H_\bullet[f]$ the f-sublevel set persistent homology without refering explicitely to a degree.

Single-parameter persistence and barcodes

In the single parameter case, all persistence modules are build from interval persistence modules.

Theorem 2 (Crawley-Boevey [4])**.** *Let $n = 1$ and M a persistence module on $\mathcal{P}$ then there exist a multiset of intervals $\mathbf{B}(M)$ such that*

$$M \simeq \bigoplus_{I \in \mathbf{B}(M)} \mathbf{k}^I$$

with this decomposition being unique up to reordering the terms of the direct sum.

The multiset $\mathbf{B}(M)$ is called the *barcode* of M. Moreover, there exist matrix reduction algorithms and software implementation to compute the barcode of a persistence module.

Unfortunately, the sweat-dream of finding an barcode-like invariant has been proved unreachable [2]. In particular, combinatorial invariants will be incomplete : two different persistent modules can share the same invariant.

Projected barcode: how to return to the single parameter-case

Several options have been envisioned to obtain combinatorial incomplete invariants. Those are based on decompositions, characteristics of the persistence modules ect... Here, we will leverage your knowledge by providing an invariant that is based on a family of reductions to the single parameter case.

The *projected barcode* presented bellow arise from an analogue viewpoint on multiple parameter persistence : sheaf theory on manifolds. This perspective is rather heavily theoretical but allows to apply long standing methods from geometry to manipulate multiple parameter persistence modules. A construction that we will here call *projection*[3], allows to obtain a collection of single-parameter persistence module. This leads a family of barcodes that can be further processed with usual methods describe earlier/latter in this course.[4]

In the context of sublevel set persistent homology the gamma linear projected barcode identifies with the following construction that we present here as a definition. For any linear form $u : \mathbb{R}^n \to \mathbb{R}$ such that for all $x \in (0, +\infty)^n$ the image $u(x)$ is positive[5].

[3] In fact it is the right derived exceptional pushforward in the derived category of sheaves

[4] One may think about restricting persistence modules to obtain the same phenomenon, this idea is fruitful [3, 7] and surprisingly is already contained in such projection [1].

[5] This condition ensures that the projected barcode is generically well-behaved e.g. continuous in u.

Definition 3. We call *gamma linear projected barcode* along u the barcode of the single-parameter persistence module $H_d[u \circ f]$.

In particular any software/algorithm that manage to compute barcodes can compute the gamma linear projected barcode.

So far, this invariant is still not combinatorial. Indeed one need to compute a continuous family of gamma linear projected barcodes. Under some further finiteness conditions one can demonstrate that a finite number of linear forms suffice to encode the entire information of the gamma-projected barcodes [5].

References

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